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Sequence of Schwarz Christoffel Mappings for a Triangulation of a Polygonal Domain *

R. E. PANTA PAZOS † & M. J. M. NEGRÓN ‡

Abstract

The aim of this paper is the generation of a sequence of Schwarz Christoffel mappings associated to a triangulation of a polygonal domain. For that we present a basic algorithm, such that for each elementary triangle of the polygonal decomposition it is defined a Schwarz Christoffel mapping.

1 Introduction

The conformal mapping of a given domain onto another is not obtained in a simpler way. The existence of such a mapping for a simply connected region $G \neq \mathbf{C}$ is guaranteed by Riemann's Mapping Theorem [2]. The determination of the analytic function giving the transformation is not an easy task. Indeed, a simply connected region can be transformed by means of elementary functions

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† Department of Mathematics, UNISC, Av. Independência, 2293, Bulding 53, room 5346, CEP 96815-900 - Santa Cruz do Sul, RS, Brasil, rpazos@unisc.br
‡ Department of Mathematics, UNISC, Av. Independência, 2293, Bulding 13, room 1301, CEP 96815-900 - Santa Cruz do Sul, RS, Brasil, pepe@unisc.br

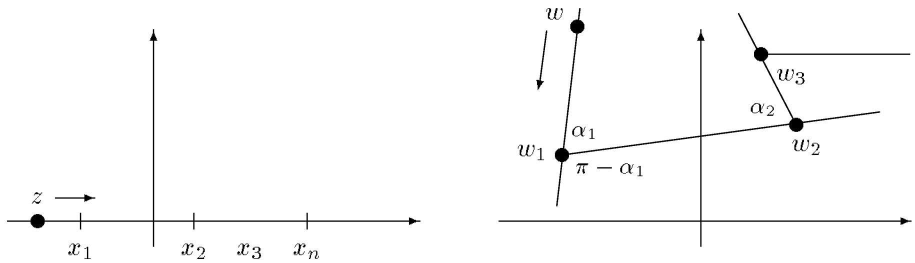

Fig 1: Mapping of the half plane **H** onto a polygonal domain, in the *w-plane*

in order to get a conformal mapping onto the open unit disk, but the unique class of domains with practical conformal representation of interest are bounded polygons. For such regions, it can be established a conformal mapping with the complex upper half plane H. This outstanding mapping was discovered independently in 1865 by the German and Swiss mathematicians H. A, Schwarz and E. B. Christoffel, respectively.

Let be the closed polygonal domain with interior angles $\alpha_1\,\pi$, $\alpha_2\,\pi$, ..., $\alpha_n\,\pi$. The Schwarz Christoffel mapping is defined by the expression [1, 3]

$$w = A + B \int_{z_0}^{z} \frac{1}{\left(\zeta - x_1\right)^{\beta_1}} \frac{1}{\left(\zeta - x_2\right)^{\beta_2}} \cdots \frac{1}{\left(\zeta - x_n\right)^{\beta_n}}\, d\zeta\,, \tag{1.1}$$

where $\beta_1 = 1 - \alpha_1$, $\beta_2 = 1 - \alpha_2$, ..., $\beta_n = 1 - \alpha_n$ are the fractions (with respect to π) of the suplementary of the given angles, and x_1, x_2, ..., x_n is an increasing set of real numbers. As z goes along the real axis of the *plane* *z*, the point w is moved through the boundary of the poligonal region which interior angles are $\alpha_1\,\pi$, $\alpha_2\,\pi$, ..., $\alpha_n\,\pi$. From numerical point of view, the mapping is reached if the lengths of each side of the polygon, and its angles, have the right values.

The integral is carried out along any path in H that joints z_0 to z, and the principal branch is used for the multiple-valued function $\left(\zeta - x_j\right)^{\beta_j}$ in the integrand such that $0 < \arg\left(\zeta - x_j\right) < \pi$, with $j = 1\,,\ 2\,,\ \ldots\,,\ n$.

Table 1. Decomposition of a Schwarz-Christoffel mapping.

Map	Analytical Expression	Geometric transformation in the $w-$plane
f_{SC}	$\displaystyle\int_{z_0}^{z} \prod_{j=1}^{n} \frac{d\zeta}{\left(\zeta - x_j\right)^{\beta_j}}$	Maps the half plane H in a polygonal region
g_{lin}	$A + B\,z$	Traslation (coef. A) Rotation and dilation (or contraction) (coef. B)

The polygon determined by f_{SC} is similar to the polygon whose angles were given and the coefficients of g_{lin} can be determined in such a way that both polygons be congruent.

The Schwarz-Christoffel formula defined by (1.1) is a tool that allows to solve boundary value problems envolving polygonal domains.

Two conditions are required for the polygonal region be closed:

$$\textbf{(a)} \quad |\beta_j| < 1 \quad \text{for all } j = 1\,, \ldots\,, n \qquad \textbf{(b)} \quad \sum_{j=1}^{n} \beta_j = 2\,, \tag{1.2}$$

where $\beta_1\,,\ \beta_2\,,\ \ldots\,,\ \beta_n$ are the exterior angles of the given polygonal region.

Proposition. 1.1 *Let be $x_1 < x_2 < \ldots < x_n$ a finite sequence of real numbers. The integral defined by f_{SC} in the Schwarz-Christoffel formula (1.1) remains bounded in the neighborhood of the singularities x_j , $j = 1, \ldots, n$ and remains bounded as $z \to \infty$. Furthermore, the mapping defined by the expression $g_{lin}(f_{SC}(z)) = A + B \displaystyle\int_{z_0}^{z} \prod_{j=1}^{n} \frac{d\zeta}{\left(\zeta - x_j\right)^{\beta_j}}$ is conformal.*

We can rewrite the expression (1.1) in the following way

$$\begin{aligned} w = &\quad A + B \int_{z_0}^{z} \zeta^{-(\beta_1 + \cdots + \beta_n)} \prod_{j=1}^{n} \left(1 - \frac{x_j}{\zeta}\right)^{-\beta_j} d\zeta \\ = &\quad A + B \int_{z_0}^{z} \frac{1}{\zeta^2} \prod_{j=1}^{n} \left(1 - \frac{x_1}{\zeta}\right)^{-\beta_j} d\zeta\,. \end{aligned} \tag{1.3}$$

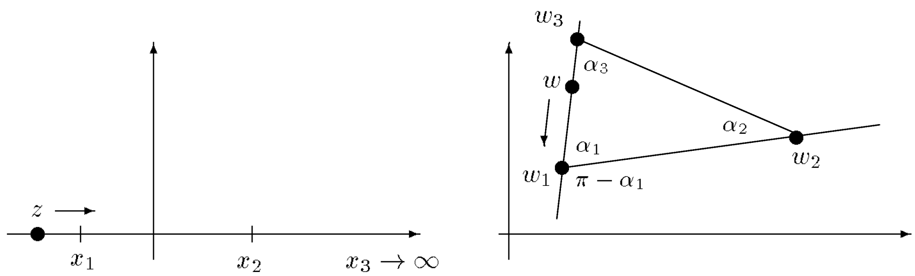

Fig 2: The z plane (left) and the triangle with vertices w_1, w_2 and w_3 (right).

proving that the integral is convergent for $z \to \infty$.

Furthermore, the SC mapping is conformal because its derivative is not null for all $z \in \mathbf{H}$, *i. e.* $B \prod_{j=1}^{n} (z - x_j)^{-\beta_j} \neq 0$. □

2 Schwarz-Christoffel Mapping for a Triangle

In order to transform the open half plane $\mathbf{H}$ on the triangle $\triangle w_1 w_2 w_3$ whose exterior angles are $\beta_1 \pi$, $\beta_2 \pi$ and $\beta_3 \pi$, such that the vertex w_1 corresponds to x_1, w_2 to x_2 and w_3 to $x_3 = \infty$, we obtain the reduzed formula

$$w = A + B \int_{z_0}^{z} \frac{1}{(\zeta - x_1)^{\beta_1}} \frac{1}{(\zeta - x_2)^{\beta_2}} \, d\zeta \,. \tag{2.4}$$

General Case. For the triangle (see Figure 2) with vertices w_1, w_2 and w_3, corresponding to x_1, x_2 and ∞, respectively, in the real axis of z *-plane*, the coefficients are $A = w_1$ and $B = \dfrac{w_2 - w_1}{\int_{x_1}^{x_2} (\zeta - x_1)^{-\beta_1} (\zeta - x_2)^{-\beta_2} \, d\zeta}$.

Special Case. For the triangle with the side $\overline{w_1 w_2}$ coinciding with the interval $[0, 1]$, being the points $w_1 = 0$, $w_1 = 1$ and w_3, corresponding to $x_1 = 0$, $x_1 = 1$ and x_3 identified with ∞, respectively on the real axis of the *z-plane*, the given transformation (2.4) is written as

$$w = B \int_0^z \frac{1}{\zeta^{\beta_1}} \frac{1}{(1-\zeta)^{\beta_2}} \, d\zeta \,, \quad \text{with} \quad B = \frac{1}{\displaystyle\int_0^1 \zeta^{-\beta_1} (1-\zeta)^{-\beta_2} \, d\zeta} \,. \tag{2.5}$$

3 Triangulation of a Polygonal Domain and a Sequence of SC Mappings

Let be a polygonal domain defined by the set $W = \{ w_0 , w_1 , \ldots , w_{n+1} \}$, which are points in the *w-plane*. Consequently we can define the following set of triangles

$$T_k = \triangle \, w_0 \, w_k \, w_{k+1} \,, \qquad \text{for all } k = 1 \ldots n \tag{3.6}$$

Suppose that each triangle is not degenerated (all the interior angles are different of zero), and choose an increasing finite sequence of real numbers (points on the real axis of the *z-plane*), *i.e*:

$$x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n \tag{3.7}$$

Next, we define a finite set of Schwarz Christoffel mappings associated to each triangle in the decomposition of the polynonal domain:

$$f_k : C \to T_k \,, \quad \text{defined by } f_k(z) = A_k + B_k \int_{x_0}^z \frac{1}{(\zeta - x_0)^{\beta_{k,1}}} \cdot \frac{1}{(\zeta - x_k)^{\beta_{k,2}}} \, d\zeta \,, \tag{3.8}$$

where $\beta_{k,1}$ and $\beta_{k,2}$ are the exterior angles of the adjacent side $\overline{w_0 \, w_k}$ of the triangle T_k . The n parameters A_k and B_k of each SC mapping f_k are obtained such as the General Case for SC mapping for a triangle:

$$A_k = w_0 \,, \qquad \text{and} \qquad B_k = \frac{w_k - w_0}{\displaystyle\int_{x_0}^{x_k} (\zeta - x_0)^{-\beta_{k,1}} \cdot (\zeta - x_k)^{-\beta_{k,2}} \, d\zeta} \,, \tag{3.9}$$

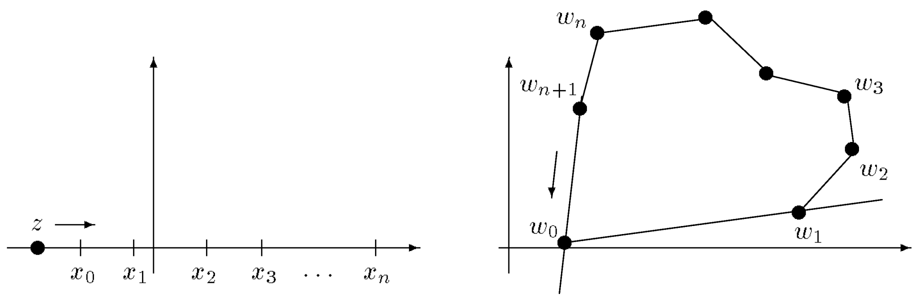

Fig 3: Polygonal domain to be decomposed in elementary triangles.

The complete script is the following one:

Algorithm of a sequence of SC mappings for a triangulation of a polygonal domain

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☐ Read the points $P_k = (a_k \, , \, b_k)$ of the polygonal domain, and define the complex numbers $w_k = a_k + i\, b_k$.

☐ Choose an increasing finite sequence of real numbers (points on the real axis of the *z-plane*), *i.e*:

$$x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n \, .$$

☐ Decompose the polygonal domain in the triangles $T_k = \triangle\, w_0\, w_k\, w_{k+1}$, and evaluate the exterior angles $\beta_{k,1}$ and $\beta_{k,2}$ which are adjacent to the side $\overline{w_0\, w_k}$ of the triangle T_k .

☐ Define the finte set of SC mappings f_k defined by the equation (3.8), and evaluate the paramenters A_k and B_k , given by (3.9).

The implementation in a computer algebraic system is straightforward. In the next section, we give some results.

4 Results with an Algebraic Computer System

For illustration we give the following example. These results were obtained with Mathcad (and Maple). The running of the worksheets are not expensive, the time employeed with Mathcad was only less than 30 seconds and with Maple it has consumed greater time. The use for the treatment of data of diverse scientific or technical areas can be oriented with data files that the above algebraic computer systems employ efficiently. But the implementation of the algorithm in an electronic worksheet is a future task for us.

Example 1. Let be the vertices $(2,1)$, $(4,2)$, $(6,4)$, $(5,7)$, $(1,6)$, $(-1,3)$ and $(-1,2)$ of a polygonal domain Ω.

First Step: We define the complex numbers:

$w_0 = 2+i\,,\ w_1 = 4+2\,i\,,\ w_2 = 6+4\,i\,,\ w_3 = 5+7\,i\,,\ w_4 = 1+6\,i\,,\ w_5 = -1+3\,i$

and $w_6 = -1+2\,i$.

Second Step: We choose the increasing real numbers:

$$x_0 = 0\,,\ x_1 = 1\,,\ x_2 = 3\,,\ x_3 = 5\ \ x_4 = 8\,,\ x_5 = 10\,.$$

Third Step: The decomposition of the polygonal region is:

$$\begin{bmatrix} T_1 = \triangle\, w_0\, w_1\, w_2 \\ T_2 = \triangle\, w_0\, w_2\, w_3 \\ T_3 = \triangle\, w_0\, w_3\, w_4 \\ T_4 = \triangle\, w_0\, w_4\, w_5 \\ T_5 = \triangle\, w_0\, w_5\, w_6 \end{bmatrix} \quad \text{such that} \quad \Omega = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5$$

Fourth Step: the respective parameters A_k and B_k ($k = 1, \ldots, 5$) for the sequence of SC mappings associated for each triangle T_k of the decompositon of Ω are described in the following table.

Table 2. Sequence of the Schwarz-Christoffel mappings.

Triangle	Analytical expression of the SC mapping
T_1	$2+i+\big(-0,148-0,0211\,i\big)\displaystyle\int_0^z \frac{d\zeta}{\zeta^{0,9428}\,\big(\zeta-1\big)^{0,8976}}$
T_2	$2+i+\big(-0,8921+0,6176\,i\big)\displaystyle\int_0^z \frac{d\zeta}{\zeta^{0,8524}\,\big(\zeta-1\big)^{0,6024}}$
T_3	$2+i+\big(-0,5816+1,3814\,i\big)\displaystyle\int_0^z \frac{d\zeta}{\zeta^{0,7896}\,\big(\zeta-1\big)^{0,2744}}$
T_4	$2+i+\big(-0,9846+0,6564\,i\big)\displaystyle\int_0^z \frac{d\zeta}{\zeta^{0,75}\,\big(\zeta-1\big)^{0,25}}$
T_5	$2+i+\big(-0,6403-0,2668\,i\big)\displaystyle\int_0^z \frac{d\zeta}{\zeta^{0,9152}\,\big(\zeta-1\big)^{0,3128}}$

5 Conclusion

We have presented a general algorithm to establish a sequence of SC mappings for a triangulation of a polygonal domain. The computational effort is not expensive. It can be applied to the storage of geographical data, such as maps.

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